The Price and Variety Effects of Horizontal Mergers

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We consider a general industry model with an arbitrary number of competitors, each offering a subset of a given line of products. The products are procured from a set of suppliers, and each product is differentiated by the supplier it is sourced from and the retailer it is sold at. The retailers engage in price competition for their entire product assortment. Our paper addresses the effects on prices, profits, and product variety due to horizontal mergers of some or all of the retailers. In the case of a full merger of all retailers, we prove that all prices and firm profits increase while the product variety *decreases*. Under a partial merger, we show that prices and firm profits continue to increase, but product variety may change in various ways. In the latter case, we provide easily verified sufficient conditions for an *expansion* of the product variety.

1. Introduction and Summary

For roughly seventy-five years, policymakers, economists, and operations managers, not to mention corporate boards, have debated and analyzed the effects of *horizontal mergers* within an industry. The U.S. government regularly publishes "merger guidelines," which identify the procedures and enforcement practices the Department of Justice (DOJ) and the Federal Trade Commission (FTC) must use to investigate whether mergers violate various antitrust laws pertaining to the Sherman Act, the Federal Trade Commission Act, and the Clayton Act, among others. Section 7 of the Clayton Act, for example, prohibits mergers or acquisitions when the effect of such a merger may be substantially to lessen "competition." In this context, "competition" is defined as a multi-dimensional phenomenon, a process that incentivizes firms to offer lower prices, enhance quality and resiliency and expand choice, among other benefits. Mergers can lessen competition when they reduce the intensity with which firms compete, as well as by reducing the number of alternative products or services that are offered in the market. And, of course, the multiple effects of a merger are of central interest to companies contemplating such mergers.

The current Biden administration has taken a particularly active role in the monitoring of potential mergers. In 2023, it issued new guidelines replacing existing ones for horizontal mergers from 2010 and for vertical mergers from 2020. As examples of recent lawsuits in this area, we mention the FTC's suit against Kroger, the country's largest supermarket operator with $150 billion in annual revenue. Kroger attempted to acquire Albertsons, the second-largest chain with $72 billion in yearly revenue. The FTC charged that the proposed deal would lead to higher prices and also narrow consumers' choices, among other negative consequences; this is in a supermarket industry in which 60% of grocery sales are concentrated among no more than five food corporations. In April 2024, the FTC sued to block the $8.5 billion merger between luxury handbag maker Tapestry (the parent of Coach, among other brands) and Capri Holdings (the owner of Michael Kors, inter alia). In January 2024, it successfully blocked Jetblue's takeover of Spirit Airlines; Attorney General Garland stated that "today's ruling is a victory for tens of millions of travelers who would have faced higher fares and fewer choices."[1] Finally, in April 2024, the DOJ, joined by 16 other state attorneys general and district attorneys, sued Apple for monopolizing the smartphone industry, once again claiming that through their judicial actions, the government was protecting customers from "higher prices and fewer choices."

The problem in predicting the effects of mergers is that until now, no models for oligopoly markets exist through which the effects of price, profit, and, in particular, product variety can be assessed numerically. Better yet, one would like a multi-product oligopoly model in which the assumed merger effects can be proven or disproven. The objective of this paper is to provide such a model and investigate the above effects of horizontal mergers.

The industry model we employ is the one introduced in Federgruen and Hu (2016), which is ideally suited for this analysis for the following reasons:

(i) The model accommodates an arbitrary number of firms engaged in price competition, each offering an arbitrary potential assortment of products;

(ii) The demand for any given product may depend on the prices charged for some or all of the products potentially sold in the market in accordance with general asymmetric customer preferences, allowing for a general combination of direct and cross-price elasticities.

(iii) The model specifies an equilibrium product assortment sold in the market as a subset of the full collection of potential products, along with associated demand volumes, and is therefore ideally suited to analyze the product variety effects of mergers.

[1] https://www.justice.gov/opa/pr/justice-department-statements-district-court-decision-block-jetblues-acquisition-s

(iv) The model has a guaranteed pure Nash equilibrium; when there are multiple equilibria, one stands out in terms of predictability.

(v) Under this special equilibrium, we obtain a set of indirect equilibrium demand functions that are analytical in the primitives of the model (subject to the computation of a single linear program possibly, with the number of variables and constraints given by the number of products).

We note that other classical consumer choice models, such as the Multinomial Logit model or any of its numerous variants, fail to predict the variety effects because in these models *all* potential products are always offered to the market, irrespective of the prevailing prices and cost structures and irrespective of the industry structure.

Here are our main results:

(1) The model allows us to evaluate the effects of any merger on a numerical basis, with computation of equilibrium prices, product assortment, sales quantities, and profits, all performed easily by evaluating analytical functions and solving possibly a single linear program of the above-mentioned size.

(2) Horizontal mergers always increase all equilibrium prices, as widely conjectured in the literature and policy papers. This is before any cost synergies that may be passed onto the consumers.

(3) Horizontal mergers always increase the profits of all firms in the industry, both the merged firm and all of its competitors.

(4) However, as to the effects on product variety, the picture is less monolithic. We prove that full consolidation into a monopoly has the effect of narrowing the equilibrium assortment, as widely assumed. But somewhat surprisingly, a partial merger of some of the competing firms may, provably, lead to an *expansion* of the equilibrium product assortment. We provide simple, sufficient conditions for this phenomenon, which can be easily verified. This phenomenon is because increased equilibrium market prices after a horizontal merger can allow more products to survive in the market when the merged firm does not prefer to shrink its product assortment. We also provide numerical examples showing various possible effects on the equilibrium product choices.

The remainder of this paper is organized as follows. Section 2 provides a brief review of the literature. Section 3 reviews our model and introduces the notation. Section 4 starts with the special case where there is a complete merger among all retailers. We treat this special case separately because, in this case, all effects on prices, profits, and product variety are unequivocal; moreover, this special case is also needed in some of the proofs pertaining to general mergers of part of the retailers. Section 5 proves our results about equilibrium prices and profits. The effects on product variety are covered in Section 6.

2. Literature Review

The early strategy works, for example, Steiner (1975), postulate that (horizontal) mergers should increase the aggregate profits of the merging firms, even in the absence of any cost efficiencies resulting from economies of scope or scale. It was also conjectured that horizontal mergers should result in an increase in equilibrium prices for *all* of the products offered in the industry. This was assumed, additionally, in the classical paper by Williamson (1968).

However, early attempts to substantiate these conjectures, e.g., by Szidarovszky and Yakowitz (1982), Salant et al. (1983), and Davidson and Deneckere (1984), all concluded from their analyses that aggregated profits of merging firms actually decline, unless of course accompanied by significant cost efficiencies due to synergies or economies of scope. A first step towards resolving this apparent enigma was provided by Deneckere and Davidson (1985). These authors explained that the above counterintuitive results were the result of analyzing the merger in the context of Cournot quantity competition models. Deneckere and Davidson (1985) proceeded to show that under (Bertrand) competition, the anticipated effects can be demonstrated: Their analysis is based on a model with completely symmetric firms and linear demand functions. (This model was first proposed in Shubik and Levitan 1980.) In their appendix, the authors extend their results to competition models with non-linear demand functions satisfying five assumptions, the most important of which is that the industry is symmetrically differentiated, i.e., all firms share the same cost structure, and price and cross-price elasticities are identical. Federgruen and Pierson (2011) analyze a model in which all profit functions of all firms (before the merger), as well as the profit functions after a full merger of all firms, are supermodular. Each of the pre-merger profit functions is also assumed to be quasi-concave in its own price variable. The authors generalize the effects proven in Deneckere and Davidson (1985).

In spite of the restrictions associated with Cournot competition, many papers in the operations management literature focus on supply chains where, at each tier, firms face Cournot competition for a single item. A classical paper in this area is Corbett and Karmarkar (2001) addressing a multi-tier supply chain. Cho (2014) addresses the same model. Moreover, Dong et al. (2024) and Wang et al. (2024) consider an industry with three firms offering a single identical product under possibly different capacity levels.[2] The firms engage in price competition so that the firm with the

[2] In general, price competition under endogenized capacity levels in a two-stage game results in quantity competition outcomes, see, e.g., Kreps and Scheinkman (1983).

lowest price attracts all uncommitted customers. Both papers address three possible mergers in this industry.

The most important restriction of the above-mentioned literature is that it assumes that every firm sells a single product, and all are in the market, regardless of prices charged or any mergers occurring. As explained in the introduction, one of the major effects we analyze is the effects on product variety, which, of course, cannot be investigated in the context of those models.

There are other related operations management papers with more detailed operational features. Xiao (2020) considers mergers in a single-tier industry, with symmetric firms having Cournot competition. She introduces the complication of proportional yield uncertainty in determining each firm's actual production quantity. Other variants of this multi-tier Cournot competition model were pursued more recently by Cho (2014) and Bimpikis et al. (2019). The latter pursues Cournot competition in networked markets where firms compete in multiple markets, with competition in each market being of the Cournot type. See also the many references in the above papers.

Cho and Wang (2017) analyze the effects of horizontal mergers when firms face a single season random demand, as in a newsvendor setting. The demands for the suppliers are normal with a given variance-covariance matrix. The mean demand of any given supplier is given by one of the special linear and symmetric functions used by Deneckere and Davidson (1985), and each firm faces an identical constant purchase price. The firms compete with each other in terms of the retail price they choose and the safety stocks they purchase at the beginning of the season. Thus, mergers have the effect of lessening price competition but also inducing cost savings for the merging firm due to risk pooling and savings in safety stock investments. Thus, the aggregate effects on retail prices can vary, unlike those in Deneckere and Davidson (1985), which do not consider situations with a risk pooling effect. The pre-merger model is that of Zhao and Atkins (2008). In contrast to these operations papers, we focus on the effects of a horizontal merger, in particular, the product variety effects, in a general industry.

3. Model and Notation

Assume an industry with I firms. For all $i \in \mathcal{I} = \{1, \ldots, I\}$, let $\mathcal{N}(i)$ denote the set of (potential) products firm i brings to the market, with (i, k) representing the k-th product in this set. Let $N \equiv |\sum_{i \in \mathcal{I}} \mathcal{N}(i)|$ denote the total number of potential products in the market. We use the following notation:

w_{ik} = the procurement cost rate for product (i, k), otherwise referred to as the "wholesale price,"

p_{ik} = the retail price charged by retailer i for product k,

d_{ik} = the consumer demand for product (i,k).

The consumer demand is based on, but not fully determined by, affine functions:

$$q(p) = a - Rp. \tag{1}$$

Here, R is an $N \times N$ matrix, and $a \geq 0$, indicating that all products are choices with non-negative demand, at least when offered for a low enough retail price. We make the following assumption about the matrix R:

ASSUMPTION 1. *The matrix R is a ZP-matrix, i.e., it is positive definite and a Z-matrix.*

(A Z-matrix has non-positive off-diagonal elements.) The affine functions in (1) describe the true demand functions only when $q(p) \geq 0$, i.e., on the *price polyhedron* $P = \{p \geq 0 \mid a - Rp \geq 0\}$. We need to extend the demand functions beyond the price polyhedron. Based on a suggestion by Shubik and Levitan (1980), we require the full demand functions to be regular with regularity defined as follows:

DEFINITION 1 (Regularity). A demand function $D(p): \mathbb{R}_+^N \to \mathbb{R}_+^N$ is said to be regular if for any product l and any price vector p, $D_l(p) = 0$ implies that $D(p + \Delta \cdot e_l) = D(p)$ for any $\Delta > 0$, where e_l denotes the l-th unit vector.

In other words, when under a given price vector p, a particular product $l = (i,k)$ is driven out of the market, any increase in the product's price has no impact on any of the demand volumes. Soon et al. (2009) showed that there is a *unique* extension of $q(\cdot)$ which is regular. For any $p \in \mathbb{R}_+^N$, a set of price corrections $t \geq 0$ needs to be applied, such that

$$d(p) = q(p-t) = a - R(p-t) \geq 0, \tag{2}$$

$$t^\top [a - R(p-t)] = 0, \quad \text{and} \quad t \geq 0. \tag{3}$$

Thus, t is the solution of a Linear Complementarity Problem (LCP), which is unique since the R matrix is a ZP-matrix, see Assumption 1. It can be determined by solving a Linear Program (LP) with N variables $t \geq 0$ and N constraints: $\min \pi^\top t$ s.t. (2) with $\pi > 0$ an arbitrary vector of positive coefficients. The vector $p' \equiv p - t$ is referred to as the projection of p onto the polyhedron P.

An alternative foundation for the demand model (2) assumes a representative consumer determining its consumption values by solving the following utility maximization problem:

$$\text{(QP)} \quad \max_{d \geq 0} (R^{-1}a - p)^\top d - \frac{1}{2} d^\top R^{-1} d. \tag{4}$$

Under Assumption 1 (in particular, the fact that R is positive definite), the utility maximization is well defined, with a unique solution, given by the LCP (2)-(3). This demand model is very popular, see, e.g., Candogan et al. (2012). (These authors consider a monopolist selling a *single* item to a network of consumers, whose utility functions are all of the type (4), with d the vector of consumption levels of the consumers. The monopolist offers a differentiated vector of cost prices w, and the consumers engage in a non-cooperative game to determine their resulting consumption values, given the vector w. This game is then embedded in a search for the vector w, which maximizes the monopolist's profits). Candogan et al. (2012, Assumption 1) assume that the R matrix is diagonally dominant (i.e., $R_{ii} > \sum_{j\neq i} R_{ij}$ for all i), a very special case of positive definiteness. This paper was, in turn, motivated by Ballester et al. (2006), employing the same (network) utility model, and was extended to the case of sequential consumption by Zhou and Chen (2018).

Federgruen and Hu (2016) characterize the equilibrium behavior in the price competition game in which each firm i attempts to maximize its overall profits $\sum_k (p_{ik} - w_{ik}) d_{ik}(p)$. To do so, a minor regularity condition is required, which the authors refer to as the NPW assumption. Here, we adopt a strong sufficient condition for the latter, imposing a limited type of symmetry on the matrix R (Federgruen and Hu 2015, Proposition 3).

Assumption 2. *The matrix R is intra-firm symmetric, i.e., $R_{ik,ik'} = R_{ik',ik}$ for all $i = 1,\ldots,I$ and $k,k' \in \mathcal{N}(i)$.*

In the special case where each firm sells a single product, the condition is trivially met. (Existing economics papers have confined themselves to this case.) The equilibrium behavior in the (Bertrand) price competition game depends heavily on the cost structure as represented by the vector w. Along with the retail price polyhedron P, we define a wholesale price polyhedron W by $W \equiv \{w \geq 0 \mid \Psi(R)q(p) = \Psi(R)a - \Psi(R)Rw \geq 0\}$, with an interior $W^o = \{w \geq 0 \mid \Psi(R)a - \Psi(R)Rw > 0\}$. Here, $\Psi(R) = T(R)[R + T(R)]^{-1}$, where

$$T(R) = \begin{pmatrix} R^\top_{\mathcal{N}(1),\mathcal{N}(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R^\top_{\mathcal{N}(I),\mathcal{N}(I)} \end{pmatrix}.$$

Under Assumption 2, $T(R)$ is symmetric, and $\Psi(R) \geq 0$. This implies that under Assumption 2, $W \supseteq P$.

If $w \in W^o$, the interior of W, there is a unique Nash equilibrium, which resides in P^o, the interior of P. If $w \in \mathbb{R}^N_+ \setminus W^o$, there is always a Nash equilibrium, but there may be multiple, infinitely many, pure equilibria. However, there is always an equilibrium which stands out in a precise way, summarized below:

Proposition 1 (Federgruen and Hu 2021, Proposition 1). *The price competition model has a pure Nash equilibrium* $(p^*|w)$ *with the following expression:*

$$(p^*|w) = \begin{cases} p^*(w) = w + [R+T(R)]^{-1}q(w) & \text{if } w \in W, \\ p^*(w') = w' + [R+T(R)]^{-1}q(w'), \text{ with } w' = \Theta(w) & \text{if } w \in \mathbb{R}^N_{++} \setminus W, \end{cases} \tag{5}$$

where $\Theta(w)$ *denotes the projection of* w *onto the polyhedron* W *along the coordinate axes.*

The above-described equilibrium $(p^*|w)$ is either unique (if $w \in W^o$) or stands out as the only one that has *global robust stability.* This means that the market converges to this equilibrium from an arbitrary starting position as a result of a plausible, iterative adjustment process. The most commonly used such an adjustment process is a so-called tâtonnement scheme, first introduced by Cournot (1838). Furthermore, as argued in Federgruen and Hu (2021), an adjustment process is all the more plausible if the firms' (iterative) adjustments can be made with limited private information only (i.e., when each firm only needs to know the demand functions of its own products and its own cost structure).

The robust best response mapping is defined as the conventional "best response" mapping but with a slight modification. For any firm $i = 1, \ldots, I$, define the robust best-response mapping as:

$$\arg\max_{p_{\mathcal{N}(i)} \geq 0} \Big\{ (p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^\top [\ (a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)}) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)}\ ] : \\ (a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)}) - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} \geq 0 \Big\}. \tag{6}$$

Thus, by solving (6), every firm ensures that its best response $p_{\mathcal{N}(i)}$ to the vector $p_{-\mathcal{N}(i)}$ has $[d(p)]_{\mathcal{N}(i)} = [q(p)]_{\mathcal{N}(i)} \geq 0$. To simplify the notation, note that the dependence of firm i's robust best response on the competitors' prices is fully determined by the $|\mathcal{N}(i)|$-dimensional vector $\alpha = a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p_{-\mathcal{N}(i)}$. Thus, define the best-response mapping as

$$RB_i(p_{-\mathcal{N}(i)}) = \arg\max_{p_{\mathcal{N}(i)} \geq 0} \{ (p_{\mathcal{N}(i)} - w_{\mathcal{N}(i)})^\top (\alpha - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)}) : \alpha - R_{\mathcal{N}(i),\mathcal{N}(i)} p_{\mathcal{N}(i)} \geq 0 \}. \tag{7}$$

Note that any firm i only needs to know the demand functions of its own products, as well as the prices charged by its competitors, to execute the best response mapping $RB_i(p_{-\mathcal{N}(i)})$. Combining these best response mappings into a single best response mapping, we get

$$RB : \mathbb{R}^N \to \mathbb{R}^N, p \mapsto RB(p) = \left(RB_1(p_{-\mathcal{N}(1)}), RB_2(p_{-\mathcal{N}(2)}), \ldots, RB_I(p_{-\mathcal{N}(I)}) \right).$$

Let $RB^{(n)}(\cdot)$ be the n-fold application of this mapping. Federgruen and Hu (2021) showed that the best response mapping $RB(\cdot)$ is a contraction mapping, which converges to the special Nash equilibrium in (5), irrespective of its starting point:

Lemma 1 (Federgruen and Hu 2021, Theorems 2 and 3). *The best response mapping* $RB(\cdot)$ *is a contraction mapping with* $(p^*|w)$ *in (8) as its unique fixed point.*

We can assess the equilibrium outcomes of the post-merger world analogous to the pre-merger world. Specifically, in the post-merger world, without loss of generality, the last $|\mathcal{I}^o|$ retailers merge, and the retailer set becomes $\hat{\mathcal{I}} = (\mathcal{I} \setminus \mathcal{I}^o) \cup \{1\} = \{1, \dots, I - |\mathcal{I}^o|, I - |\mathcal{I}^o| + 1\}$ and

$$\begin{aligned}
\hat{W} &= \{w \geq 0 \mid \hat{\Psi}(R)(a - Rw) \geq 0\}, \\
\hat{\Psi}(R) &= \hat{T}(R)[R + \hat{T}(R)]^{-1}, \\
\hat{T}(R) &= \begin{pmatrix}
R^T_{\mathcal{N}(1)\mathcal{N}(1)} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & R^T_{\mathcal{N}(I-|\mathcal{I}^o|+1)\mathcal{N}(I-|\mathcal{I}^o|+1)} & \cdots & R^T_{\mathcal{N}(I-|\mathcal{I}^o|+1)\mathcal{N}(I)} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & R^T_{\mathcal{N}(I)\mathcal{N}(I-|\mathcal{I}^o|+1)} & \cdots & R^T_{\mathcal{N}(I)\mathcal{N}(I)}
\end{pmatrix}.
\end{aligned}$$

Assumption 3. $\hat{T}(R)$ *is symmetric. (As a result,* $T(R)$ *is also symmetric.)*

Assumption 3 is a somewhat stronger version of Assumption 2 that is equivalent to the symmetry of $T(R)$. Under Assumption 3, applying Proposition 1 to the post-merger world, we have a pure Nash equilibrium $(p^*|w)$ with the following expression:

$$(\hat{p}^*|w) = \begin{cases} \hat{p}^*(w) = w + [R + \hat{T}(R)]^{-1} q(w) & \text{if } w \in \hat{W}, \\ \hat{p}^*(w') = w' + [R + \hat{T}(R)]^{-1} q(w'), \text{ with } w' = \hat{\Theta}(w) & \text{if } w \in \mathbb{R}^N_{++} \setminus \hat{W}, \end{cases} \tag{8}$$

where $\hat{\Theta}(w)$ denotes the projection of w onto the polyhedron $\hat{W}$ along the coordinate axes.

Let $\mathcal{N}^*$ (resp., $\hat{\mathcal{N}}^*$) denote the set of products sold in the pre-merger (resp., post-merger) world at equilibrium $(p^*|w)$ (resp., $(\hat{p}^*|w)$).

4. A Full Merger of All Retailers

In this section, we first characterize the merger effects when all firms in the oligopoly merge into a single monopoly. This case is of interest in itself and the focus of most merger models covered in the past; see the literature review. In addition, the results from this special case are needed to prove the more general case where only some of the firms merge. This more general case is the topic of the next section.

Under a full merger of all firms, Theorem 1 shows that all anticipated effects occur: (1) regardless of the cost vector w, the equilibrium prices for all products increase weakly, see part (iv); (2) for all products, the equilibrium demand levels decrease weakly, see part (v); (3) the total equilibrium profit increases weakly, after the merger, see part (vi); (4) even the equilibrium product assortment shrinks weakly after the merger, meaning that some products offered before the merger become unattainable post the merger, and no products that were unattainable before the merger enter the market, see part (ii). There are two more parts of Theorem 1, which are of a more technical nature but essential in proving the remainder.

Theorem 1. *If the merger results in a monopoly, i.e., $\hat{T}(R) = R$ under Assumption 3, then*

(i) $\hat{W} \subseteq W$.

(ii) $\hat{\mathcal{N}}^* \subseteq \mathcal{N}^*$ *(equilibrium product assortment shrinks after the merger).*

(iii) For any $w \geq 0$, $\hat{\Theta}(w) \leq \Theta(w)$.

(iv) $(p^*|w) \leq (\hat{p}^*|w)$ *for any* $w \geq 0$ *(equilibrium prices increase after the merger).*

(v) $d_l(p^*|w) \geq d_l(\hat{p}^*|w)$ *for any* $l \in \mathcal{N}$ *(equilibrium demand levels decrease after the merger).*

(vi) $\sum_i \pi_i(p^*|w) \leq \sum_i \pi_i(\hat{p}^*|w)$ *(the total equilibrium profit value increases after the merger).*

Proof of Theorem 1. (i) First, we show that $\Psi(R)$ and $\hat{\Psi}(R)$ are invertible. Note that $T(R)$ and $R+T(R)$ are invertible because they are ZP-matrices, see Horn and Johnson (1991, Theorem 2.5.3). Hence, $[R+T(R)]^{-1}$ is invertible. Moreover, by definition, $\Psi(R) = T(R)[R+T(R)]^{-1}$, and thus, $\det(\Psi(R)) = \det(T(R))\det([R+T(R)]^{-1}) > 0$, since both $T(R)$ and $[R+T(R)]^{-1}$ are invertible; thus, $\Psi(R)$ is invertible as well. Similarly, $\hat{\Psi}(R)$ is invertible. Moreover, both $\det(T(R)) > 0$ and $\det([R+T(R)]^{-1}) > 0$, since both $T(R)$ and $R+T(R)$ are positive definite matrices (see Assumption 1).

Now we show that

$$\Psi(R)[\hat{\Psi}(R)]^{-1} \geq 0. \tag{9}$$

(9) proves part (i): if $w \in \hat{W}$, $\hat{\Psi}(R)(a - Rw) \geq 0$. Under (9), we can pre-multiply both sides of this inequality with $\Psi(R)[\hat{\Psi}(R)]^{-1}(\geq 0)$ to conclude that $\Psi(R)(a - Rw) \geq 0$, i.e., $w \in W$.

To prove (9), because $\hat{T}(R) = R$,

$$\hat{\Psi}(R) = \hat{T}(R)[R + \hat{T}(R)]^{-1} = \frac{\mathrm{I}}{2}.$$

Thus,

$$\Psi(R)[\hat{\Psi}(R)]^{-1} = 2\Psi(R).$$

Now we show that

$$\Psi(R) = T(R)[R + T(R)]^{-1} \geq \frac{\mathrm{I}}{2}, \tag{10}$$

and as a result,

$$\Psi(R)[\hat{\Psi}(R)]^{-1} = 2\Psi(R) \geq \mathrm{I}, \tag{11}$$

thus proving (9).

To prove (10), since $T(R)$ is a ZP-matrix, $T(R)$ is invertible and $[T(R)]^{-1} \geq 0$, by Horn and Johnson (1991, Theorem 2.5.3). Then because $R \leq T(R)$ under Assumption 3, $R[T(R)]^{-1} \leq \mathrm{I}$ and hence,

$$R[T(R)]^{-1} + \mathrm{I} \leq 2\mathrm{I}.$$

$R[T(R)]^{-1} + \mathrm{I}$ is a ZP-matrix by Federgruen and Hu (2015, Lemma A.1 parts (c) and (d)), we thus have, since $2\mathrm{I}$ is a ZP-matrix as well by part (b) of this Lemma,

$$\Psi(R) = T(R)[R + T(R)]^{-1} = [R[T(R)]^{-1} + \mathrm{I}]^{-1} \geq \frac{\mathrm{I}}{2},$$

thus proving (10).

(ii) By Federgruen and Hu (2021, Eq. (6)) and Federgruen and Hu (2015, Eq. (8)), the pre-merger equilibrium sales volumes are given by: $\Psi(R)(a - R(w - t^*))$, and the post-merger equilibrium volumes by: $\hat{\Psi}(R)(a - R(w - \hat{t}^*))$, where t^* is the unique solution to LCP: $\Psi(R)(a - R(w - t)) \geq 0$, $t^\top[\Psi(R)(a - R(w - t))] = 0$ and $t \geq 0$, and $\hat{t}^*$ is the unique solution to LCP: $\hat{\Psi}(R)(a - R(w - \hat{t})) \geq 0$, $\hat{t}^\top[\hat{\Psi}(R)(a - R(w - \hat{t}))] = 0$ and $\hat{t} \geq 0$. Note from the complementarity conditions of the LCP that $\mathcal{N}^* = \{l \in \mathcal{N} \mid t_l^* = 0\}$ and $\hat{\mathcal{N}}^* = \{l \in \mathcal{N} \mid \hat{t}_l^* = 0\}$. To prove $\hat{\mathcal{N}}^* \subseteq \mathcal{N}^*$, it therefore suffices to show that

$$\hat{t}^* \geq t^*. \tag{12}$$

Fix $l \in \mathcal{N}$. By Mangasarian (1976, Theorem 3), the (unique) solution to the LCP: $\Psi(R)(a - R(w - t)) \geq 0$, $t^\top[\Psi(R)(a - R(w - t))] = 0$ and $t \geq 0$ is the (unique) solution of any LP of the form:

$$\min\ \pi^\top t \quad \text{s.t. } \Psi(R)(a - R(w - t)) \geq 0, \quad t \geq 0, \tag{13}$$

where π is *any* vector of positive coefficients. Similarly, the (unique) solution to the LCP: $\hat{\Psi}(R)(a - R(w - \hat{t})) \geq 0$, $\hat{t}^\top[\hat{\Psi}(R)(a - R(w - \hat{t}))] = 0$ and $\hat{t} \geq 0$ is the (unique) solution of any LP of the form:

$$\min\ \pi^\top \hat{t} \quad \text{s.t. } \hat{\Psi}(R)(a - R(w - \hat{t})) \geq 0, \quad \hat{t} \geq 0, \tag{14}$$

where π is *any* vector of positive coefficients. Select the vector π such that $\pi_l = 1$ and $\pi_{l'} = \epsilon$ for all $l' \neq l$, for some $\epsilon > 0$. In the proof of part (i), we showed that $\Psi(R)[\hat{\Psi}(R)]^{-1} \geq 0$, see (9). Then any feasible solution to LP (14) is a feasible solution to LP (13), so that $t_l^* + \epsilon \sum_{l' \neq l} t_{l'}^* = \pi^\top t^* \leq \pi^\top \hat{t}^* = \hat{t}_l^* + \epsilon \sum_{l' \neq l} \hat{t}_{l'}^*$. Thus $t_l^* \leq \hat{t}_l^*$ follows by taking the limit for $\epsilon \searrow 0$.

(iii) By the definition of the projection mappings $\Theta(w)$ and $\hat{\Theta}(w)$ (see Federgruen and Hu 2015, Eq. (8)), $\hat{\Theta}(w) = w - \hat{t}^* \leq w - t^* = \Theta(w)$, where the inequality follows from (12).

(iv) Consider a given industry structure in our oligopoly. Here, all the notation is generic, and later, we will specify the structure and apply the results we obtain in this paragraph. Recall $d(p^*|w) = Q(\Theta(w)) = q(p^*|w) \geq 0$ for any $w \geq 0$ by Federgruen and Hu (2015, Theorems 2 and 3). In view of Federgruen and Hu (2015, Eq. (8)), by the complementarity property in the LCP, if $[q(p^*|w)]_l > 0$, we must have $[\Theta(w)]_l = w_l$. This is because if this is not true, i.e., $[\Theta(w)]_l < w_l$, by the complementarity condition of the LCP, we have $[q(p^*|w)]_l = [Q(\Theta(w))]_l = 0$ since $t_l = w_l - w'_l > 0$ (note that we denote $\Theta(w) = w'$). This leads to a contradiction.

Fix $w \geq 0$. Now, consider the post-merger world. For any product $l \in \mathcal{L} = \{l \in \mathcal{N} \mid [q(\hat{p}^*|w)]_l > 0\}$, by the above argument, $[\hat{\Theta}(w)]_l = w_l$.

Now we prove that in the pre-merger world, we also have $[\Theta(w)]_l = w_l$ for any $l \in \mathcal{L}$. This can be shown by contradiction. Suppose, on the contrary, we have $[\Theta(w)]_l < w_l$. By part (i) ($\hat{W} \subseteq W$), we must have $[\hat{\Theta}(w)]_l \leq [\Theta(w)]_l < w_l$. By the complementarity proof we have above, we must have $[q(\hat{p}^*|w)]_l = 0$, which leads to a contradiction with $l \in \mathcal{L} = \{l \in \mathcal{N} \mid [q(\hat{p}^*|w)]_l > 0\}$.

Thus, we have

$$\begin{aligned}
[q(p^*|w)]_l &= [\Psi(R)(a - R\Theta(w))]_l \\
&\geq [\Psi(R)(a - R\hat{\Theta}(w))]_l \\
&= [\Psi(R)[\hat{\Psi}(R)]^{-1}\hat{\Psi}(R)(a - R\hat{\Theta}(w))]_l \\
&\geq [\hat{\Psi}(R)(a - R\hat{\Theta}(w))]_l = [q(\hat{p}^*|w)]_l,
\end{aligned}$$

where the first equality is due to $\Theta(w) \in W$ and Federgruen and Hu (2015, Theorem 3), the first inequality is due to $[\Theta(w)]_l = [\hat{\Theta}(w)]_l = w_l$, $[\Theta(w)]_{-l} \geq [\hat{\Theta}(w)]_{-l}$ by part (iii) and $\Psi(R)R$ is a Z-matrix (see Federgruen and Hu 2015, Proposition 3), and the last inequality is due to (11) and $\hat{\Theta}(w) \in \hat{W} = \{w' \geq 0 \mid \hat{\Psi}(R)(a - Rw') \geq 0\}$.

For any product l such that $[q(\hat{p}^*|w)]_l = 0$, $[q(p^*|w)]_l \geq 0 = [q(\hat{p}^*|w)]_l$ holds trivially.

Combining the two cases, we have

$$q(p^*|w) \geq q(\hat{p}^*|w). \tag{15}$$

That is,

$$a - R(p^*|w) \geq a - R(\hat{p}^*|w). \tag{16}$$

Thus, we have the desired result, by canceling a and pre-multiplying $-R^{-1}(\leq)0$ on both sides of (16).

(v) By (15),

$$d(p^*|w) = q(p^*|w) \geq q(\hat{p}^*|w) = d(\hat{p}^*|w).$$

(vi) We have

$$\sum_i \pi_i(p^*|w) \leq \sum_i \pi_i(\hat{p}^*|w)$$

because $(p^*|w)$ is a feasible solution for the monopoly after the merger, but the monopoly might strictly increase its profit value by optimizing prices. □

5. A Partial Merger of Some of the Retail Firms: Implications for Prices and Profits

In this section, we generalize the results to the most general case where only part of the retailers merge. It turns out that equilibrium prices continue to increase for all products, as well as equilibrium profits. The same categoric conclusions cannot be made for the product variety. This will be shown in the next section (Section 6) along with sufficient conditions under which this product assortment, in fact, expands.

THEOREM 2. *(i) Equilibrium prices (weakly) increase after the merger, i.e.,* $(p^*|w) \leq (\hat{p}^*|w)$ *for any* $w \geq 0$.

(ii) Equilibrium profit values increase after the merger.

Proof of Theorem 2. (i) Without loss of generality, for notational simplicity, we assume the last two retailers, $I-1$ and I, merge.

Before the merger, the robust best response mapping is

$$RB(p) = (RB_1(p_{-\mathcal{N}(1)}), \ldots, RB_I(p_{-\mathcal{N}(I)})).$$

After the merger, the robust best mapping becomes:

$$\widehat{RB}(p) = (RB_1(p_{-\mathcal{N}(1)}), \ldots, RB_{I-2}(p_{-\mathcal{N}(I-2)}), \widehat{RB}_{I-1}(p_{-\mathcal{N}(I-1)\cup\mathcal{N}(I)})).$$

By Federgruen and Hu (2021, Theorem 3), both $RB(p)$ and $\widehat{RB}(p)$ are contraction mappings. As a result, both mappings have a unique fixed point, i.e., $(p^*|w)$ and $\hat{p}^*(\hat{\Theta}(w))$, respectively.

Consider the robust best response process in the pre-merger world starting at the point $p = (p^*|w)$. The robust best response process associated with $RB(p)$ would stay at $p = (p^*|w)$ because it is the fixed point for the contraction mapping $RB(p)$ by Federgruen and Hu (2021, Theorem 2).

Now we show that:

$$\widehat{RB}(p^*|w) \geq RB(p^*|w) = (p^*|w). \tag{17}$$

First, $[\widehat{RB}(p^*|w)]_{\mathcal{N}(1)\cup\cdots\cup\mathcal{N}(I-2)} = [RB(p^*|w)]_{\mathcal{N}(1)\cup\cdots\cup\mathcal{N}(I-2)}$.

Now we compare $(RB_{I-1}((p^*|w)_{-\mathcal{N}(I-1)}), RB_I((p^*|w)_{-\mathcal{N}_I}))$ and $\widehat{RB}_{I-1}((p^*|w)_{-\mathcal{N}(I-1)\cup\mathcal{N}(I)})$. Note that given $(p^*|w)_{-\mathcal{N}(I-1)\cup\mathcal{N}(I)}$ fixed, $(RB_{I-1}((p^*|w)_{-\mathcal{N}(I-1)}), RB_I((p^*|w)_{-\mathcal{N}_I})) = ((p^*|w)_{\mathcal{N}(I-1)}, (p^*|w)_{\mathcal{N}(I)})$ because in the pre-merger world, $(p^*|w)$ is the equilibrium we target, while $\widehat{RB}_{I-1}((p^*|w)_{-\mathcal{N}(I-1)\cup\mathcal{N}(I)})$ represents the monopoly optimal prices for the merged firm, when the prices of all other firms $1,\ldots,I-2$ are fixed at their $(p^*|w)$ values. (Actually, the optimal monopoly prices for the merged firm are also price equilibrium values and can be chosen as the special equilibrium, denoted in Proposition 1, which satisfies: $\alpha_{\mathcal{N}(I-1)\cup\mathcal{N}(I)} - R_{\mathcal{N}(I-1)\cup\mathcal{N}(I),\mathcal{N}(I-1)\cup\mathcal{N}(I)} p_{\mathcal{N}(I-1)\cup\mathcal{N}(I)} \geq 0$. This means that $\widehat{RB}_{I-1}((p^*|w)_{-\mathcal{N}(I-1)\cup\mathcal{N}(I)})$ is a vector of optimal monopoly prices in the merged industry. Comparing the duopoly $\{l-1,l\}$, with the prices of all firms $1,\ldots,l-2$, fixed at their $(p^*|w)$ values, on the one hand, in the industry in which these two firms merge, we apply part (iv) of Theorem 1 to conclude that

$$(RB_{I-1}((p^*|w)_{-\mathcal{N}(I-1)}), RB_I((p^*|w)_{-\mathcal{N}_I})) \leq \widehat{RB}_{I-1}((p^*|w)_{-\mathcal{N}(I-1)\cup\mathcal{N}(I)}),$$

which demonstrates (17).

By Federgruen and Hu (2016, Theorem 4(b)), noting that each firm solves a best response problem to a given competitors' price vector as a monopoly and the intercept vector of the demand for the "monopoly" firm increases in the competitors' price vector, we have $RB_i(p_{-\mathcal{N}_i}), i = 1,\ldots,I-2$, is increasing in $p_{-\mathcal{N}_i}$ and $\widehat{RB}_{I-1}(p_{-\mathcal{N}(I-1)\cup\mathcal{N}(I)})$ is increasing in $p_{-\mathcal{N}(I-1)\cup\mathcal{N}(I)}$. Therefore, $\widehat{RB}(p)$ is increasing in p. Combined with (17), we have

$$(\hat{p}^*|w) = \lim_{n\to\infty} \widehat{RB}^{(n)}(p^*|w) \geq \widehat{RB}(p^*|w) \geq RB(p^*|w) = (p^*|w).$$

(ii) First, consider any firm i's best response problem as a monopoly's problem. Since the intercept of the raw demand function of firm i increases when the competitors choose higher equilibrium prices:

$$(0 \leq) a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} p^*_{-\mathcal{N}(i)} \leq a_{\mathcal{N}(i)} - R_{\mathcal{N}(i),-\mathcal{N}(i)} \hat{p}^*_{-\mathcal{N}(i)},$$

(which is due to R is a Z-matrix and $p^*_{-\mathcal{N}(i)} \leq \hat{p}^*_{-\mathcal{N}(i)}$ by part (i)), by applying Federgruen and Hu (2016, Theorem 4(g)), we have

$$\pi_i(RB_i(p^*_{-\mathcal{N}(i)})) \leq \pi_i(RB_i(\hat{p}^*_{-\mathcal{N}(i)})). \tag{18}$$

For those firms who are not involved in the merger, we have the desired result.

Now consider firms $I-1$ and I who are involved in the merger. We have

$$\sum_{i=I-1,I} \pi_i(RB_i(p^*_{-\mathcal{N}(i)})) \leq \sum_{i=I-1,I} \pi_i(RB_i(\hat{p}^*_{-\mathcal{N}(i)})) \leq \sum_{i=I-1,I} \pi_i(\widehat{RB}_i(\hat{p}^*_{-\mathcal{N}(i)})),$$

where the first inequality is due to (18) and the second inequality is due to the fact that $(RB_i(\hat{p}^*_{-\mathcal{N}(i)}), i = I-1, I)$, is a feasible solution for the merged firm, but the merged firm might strictly increase its profit value by choosing the best response $(\widehat{RB}_i(\hat{p}^*_{-\mathcal{N}(i)}), i = I-1, I)$ after the merger. □

6. The Product Variety Effects of a Horizontal Merger: Examples and Sufficient Conditions for Expansion of the Assortment

While the effects on equilibrium price and profits are unequivocal, as per Thereom 2, the impacts on product variety are more subtle and less categorical, compared to the situation of a full merger of the industry, see Theorem 1(ii). In this section, we address the effects of a (partial) horizontal merger on the equilibrium product assortment. We start with 3 examples, showing that the product assortment may indeed shrink, as feared by many regulators, but it may also expand or change altogether. The examples are followed by Theorem 3, which gives sufficient and easily verified conditions under which an *expansion* of the equilibrium assortment can be expected.

Example 1 (Equilibrium product assortment *expands* after the merger). Here is an example where the product variety expands due to a merger. Consider a market with three retailers $i = 1, 2, 3$, each carrying a single product $i = 1, 2, 3$. Let $a = (1, 1, 1)^T$ and $R = \begin{pmatrix} 1 & -0.4 & -0.4 \\ -0.4 & 1 & -0.4 \\ -0.4 & -0.4 & 1 \end{pmatrix}$. The matrix R is clearly a Z-matrix, and it is positive definite since all of its principal minors are positive.

Select $w^o = (2.5, 1, 1)$. In the pre-merger world, firms 1, 2, and 3 would set equilibrium prices $p^* = (p_1^*, p_2^*, p_3^*) = (2.5, 1.875, 1.875)$, receive sales volumes $d(p^*) = (0, 0.875, 0.875)$, and earn profits $\pi(p^*) = (0, 0.7656, 0.7656)$. That is, firm 1 and its product are priced out of the market because of its high procurement cost. The equilibrium product assortment before the merger is $\mathcal{N}^*(w^o) = \{2, 3\}$.

In a post-merger world where firms 2 and 3 merge, firms would set equilibrium prices $\hat{p}^* = (\hat{p}_1^*, \hat{p}_2^*, \hat{p}_3^*) = (2.6346, 2.2115, 2.2115)$, receive sales volumes $d(\hat{p}^*) = (0.1346, 0.7269, 0.7269)$, and earn profits $\pi(\hat{p}^*) = (0.0181, 0.8807, 0.8807)$. Now, since the two originally fighting competitors merged, the newly merged firm can charge higher retail prices. In accordance with Theorem 2, due to the alleviated price competition, firm 1 also benefits by entering the market and earning a positive profit. The equilibrium product assortment after the merger becomes $\hat{\mathcal{N}}^*(w^o) = \{1, 2, 3\}$. The equilibrium product assortment *expands* after the horizontal merger. □

Example 2. (Equilibrium product assortment *changes* after the merger, but the number of available products stays the same). Here is an example where the horizontal merger affects the product variety, but the number of products does not change. This example is created from the previous example by making the competition before the merger between products 2 and 3 more intense and by increasing the cost of product 3 so that after the merger, product 3 is going to be pushed out of the market after the merger.

Consider a market with three retailers $i = 1,2,3$, each carrying a single product $i = 1,2,3$. Let $a = (1,1,1)^T$ and $R = \begin{pmatrix} 1 & -0.4 & -0.4 \\ -0.4 & 1 & -0.5 \\ -0.4 & -0.5 & 1 \end{pmatrix}$.

Select $w^o = (3.5,1,3)$. In the pre-merger world, firms 1, 2, and 3 would set equilibrium prices $p^* = (p_1^*, p_2^*, p_3^*) = (3.3051, 2.4814, 3.2814)$, receive sales volumes $d(p^*) = (0, 1.4814, 0.2814)$, and earn profits $\pi(p^*) = (0, 2.1944, 0.0792)$. That is, firm 1 and its product are priced out of the market because of its high procurement cost. The equilibrium product assortment before the merger is $\mathcal{N}^*(w^o) = \{2,3\}$.

In a post-merger world where firms 2 and 3 merge, firms would set equilibrium prices $\hat{p}^* = (\hat{p}_1^*, \hat{p}_2^*, \hat{p}_3^*) = (3.6250, 2.9500, 3.9250)$, receive sales volumes $d(\hat{p}^*) = (0.1250, 1.4625, 0)$, and earn profits $\pi(\hat{p}^*) = (0.0156, 2.8519, 0)$. Now, since two originally fighting competitors merged, the newly merged firm charges higher retail prices, in accordance with Theorem 2. Due to the alleviated price competition, firm 1 also benefits by entering the market and earning a positive profit. In addition, because of the high substitutability between products 2 and 3 and the high cost of product 3, product 3 is dropped from the product line after the horizontal merger. The equilibrium product assortment after the merger becomes $\hat{\mathcal{N}}^*(w^o) = \{1,2\}$. The equilibrium product assortment *changes* after the horizontal merger, but the number of available products stays the same. □

Finally, we show an example of an industry where the product variety shrinks due to the horizontal merger. In this example, there is a full merger. Here, the shrinkage of the product variety is proven in general; see Theorem 1.

Example 3 (Equilibrium product assortment *shrinks* after the merger). Consider a duopoly in which each retailer $i = 1,2$ carries a single product $i = 1,2$. Let $a = (1,1)^T$ and $R = \begin{pmatrix} 1 & -\gamma \\ -\gamma & 1 \end{pmatrix}$, with $\gamma \in [0,1)$. We have $\Psi(R) = T(R)[R + T(R)]^{-1} = \frac{1}{4-\gamma^2} \begin{pmatrix} 2 & \gamma \\ \gamma & 2 \end{pmatrix}$ and $\hat{\Psi}(R) = \frac{\mathrm{I}}{2}$.

In the pre-merger world,

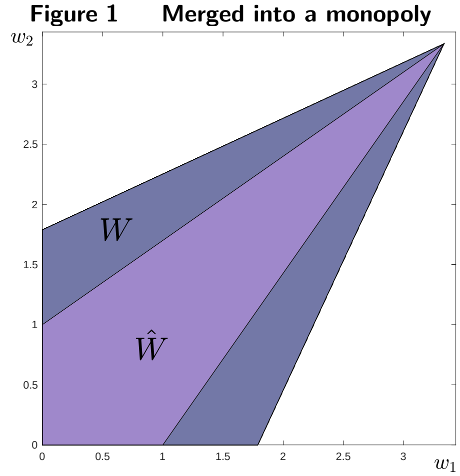
$$W = \{w \geq 0 \mid \Psi(R)a - \Psi(R)Rw \geq 0\} = \left\{ (w_1, w_2) \geq 0 \,\middle|\, \begin{array}{l} 2 + \gamma - (2-\gamma^2)w_1 + \gamma w_2 \geq 0 \\ 2 + \gamma + \gamma w_1 - (2-\gamma^2)w_2 \geq 0 \end{array} \right\}.$$

In the post-merger world,

$$\hat{W} = \{w \geq 0 \mid \hat{\Psi}(R)a - \hat{\Psi}(R)Rw \geq 0\} = \left\{ (w_1, w_2) \geq 0 \,\middle|\, \begin{matrix} 1 - w_1 + \gamma w_2 \geq 0 \\ 1 + \gamma w_1 - w_2 \geq 0 \end{matrix} \right\}.$$

For $\gamma = 0.7$, we plot W and $\hat{W}$ in the Figure 1. Indeed, it is observed that $\hat{W}$ is completely contained within W, i.e., $W \setminus \hat{W} \neq \emptyset$. Then, for any $w \in W^o \setminus \hat{W}^o$, the equilibrium product variety in the market reduces from 2 to 1 after the horizontal merger. For example, pick $w^o = (0.5, 1.5) \in W^o \setminus \hat{W}^o$. In the pre-merger world, firms 1 and 2 would set equilibrium prices $p^* = (p_1^*, p_2^*) = (1.3533, 1.7236)$, receive sales volumes $d(p^*) = (0.8533, 0.2236)$, and earn profits $\pi(p^*) = (0.7281, 0.0500)$. In the post-merger world, the monopoly would set optimal prices $\hat{p}^* = (\hat{p}_1^*, \hat{p}_2^*) = (1.9167, 2.3417)$ with the high-cost product 2 priced out of the market, receive sales volumes $d(\hat{p}^*) = (0.7225, 0)$, and earn profits 1.0235 solely from product 1. □

Figure 1 **Merged into a monopoly**



Note. $\gamma = 0.7$.

The above examples show that the effects of mergers on product variety can be of all types, contrary to common perception. (This common perception is embedded in the U.S. government's Merger Guidelines (2023)[3].) Our model allows us to quickly verify what happens in any given industry, as specified by its firm and product structure, and the primitives a, R, and w.

The next main result (Theorem 3) provides a simple set of sufficient conditions under which the offered product set after the merger $\hat{\mathcal{N}}^*(w)$ in fact expands beyond the original set $\mathcal{N}^*(w)$.

Without loss of generality, we consider a horizontal merger of two retailers, say the last two. For

[3] https://www.justice.gov/atr/2023-merger-guidelines

a merger with multiple retailers, we can first merge two retailers and then iteratively add a new firm to the merged firm, one at a time. We write the matrix R in block form:

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} R_{11} & \multicolumn{2}{c}{R_{1,2}} \\ R_{2,1} & \begin{matrix} R_{22} & R_{23} \\ R_{32} & R_{33} \end{matrix} \end{pmatrix} \equiv \begin{pmatrix} R_{11} & R_{1,2} \\ R_{2,1} & R_{2,2} \end{pmatrix}.$$

Here, indices 2 and 3 refer to the two merging firms, and 1 refers to the remainder of the industry. The following proposition is key to proving Theorem 3.

Proposition 2. *Assume, in addition to intra-firm symmetry (Assumption 3),* $R_{1,2} = R_{2,1}^{\top}$. *For any* $w \in W$, *we have*

$$[\hat{\Psi}(R)(a - Rw)]_{\overline{\mathcal{M}}} \geq 0,$$

where $\mathcal{M}$ *is the set of the products involved in the merger and* $\overline{\mathcal{M}}$ *is the set of the rest of the products.*

Proof of Proposition 2. We showed the invertibility of $\Psi(R)$ and $\hat{\Psi}(R)$ at the start of the proof of Theorem 1.

Now, we show that

$$\left[\hat{\Psi}(R)[\Psi(R)]^{-1}\right]_{\overline{\mathcal{M}},\mathcal{N}} \geq 0. \tag{19}$$

(19) proves the proposition because: For any $w \in W = \{w' \geq 0 \mid \Psi(R)(a - Rw') \geq 0\}$, under (19), we can pre-multiply both sides of $\Psi(R)(a - Rw) \geq 0$ with $[\hat{\Psi}(R)[\Psi(R)]^{-1}]_{\overline{\mathcal{M}},\mathcal{N}} (\geq 0)$ to conclude that

$$[\hat{\Psi}(R)(a - Rw)]_{\overline{\mathcal{M}}} \geq 0.$$

Write

$$\begin{aligned} \Sigma \equiv \hat{\Psi}(R)[\Psi(R)]^{-1} &= \hat{T}(R)[R + \hat{T}(R)]^{-1}[T(R)[R + T(R)]^{-1}]^{-1} \\ &= \hat{T}(R)[R + \hat{T}(R)]^{-1}[R + T(R)][T(R)]^{-1}. \end{aligned}$$

Next we will show that $\Sigma_{\overline{\mathcal{M}},\mathcal{N}} \geq 0$.

Recall

$$T(R) = \begin{pmatrix} T(R_{11}) & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{pmatrix}$$

and

$$\hat{T}(R) = \begin{pmatrix} T(R_{11}) & 0 & 0 \\ 0 & R_{22} & R_{23} \\ 0 & R_{32} & R_{33} \end{pmatrix}.$$

Denote

$$\tilde{R} \equiv \begin{pmatrix} 2R_{22} & R_{23} \\ R_{32} & 2R_{33} \end{pmatrix}$$

$$\Delta \equiv \begin{pmatrix} 0 & R_{23} \\ R_{32} & 0 \end{pmatrix} \leq 0.$$

Thus

$$T(R) = \begin{pmatrix} T(R_{11}) & 0 \\ 0 & \frac{\tilde{R}-\Delta}{2} \end{pmatrix}$$

and

$$\hat{T}(R) = \begin{pmatrix} T(R_{11}) & 0 \\ 0 & \frac{\tilde{R}+\Delta}{2} \end{pmatrix}.$$

Denote

$$\Gamma \equiv \left(R_{11} + T(R_{11}) - \frac{1}{2} R_{1,2} R_{2,2}^{-1} R_{2,1} \right)^{-1}$$

and

$$\chi \equiv \left(2R_{2,2} - R_{2,1} [R_{11} + T(R_{11})]^{-1} R_{1,2} \right)^{-1}.$$

$$\begin{aligned}
&[R+\hat{T}(R)]^{-1}[R+T(R)] \\
&= \begin{pmatrix} R_{11}+T(R_{11}) & R_{1,2} \\ R_{2,1} & 2R_{2,2} \end{pmatrix}^{-1} \begin{pmatrix} R_{11}+T(R_{11}) & R_{1,2} \\ R_{2,1} & \tilde{R} \end{pmatrix} \\
&= \begin{pmatrix} R_{11}+T(R_{11}) & R_{1,2} \\ R_{2,1} & 2R_{2,2} \end{pmatrix}^{-1} \left[\begin{pmatrix} R_{11}+T(R_{11}) & R_{1,2} \\ R_{2,1} & 2R_{2,2} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \right] \\
&= \mathrm{I} - \begin{pmatrix} R_{11}+T(R_{11}) & R_{1,2} \\ R_{2,1} & 2R_{2,2} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \\
&= \mathrm{I} - \begin{pmatrix} \Gamma & -\frac{1}{2}\Gamma R_{1,2} R_{2,2}^{-1} \\ -\frac{1}{2} R_{2,2}^{-1} R_{2,1} \Gamma & \chi \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{I} & \frac{1}{2}\Gamma R_{1,2} R_{2,2}^{-1} \Delta \\ 0 & \mathrm{I} - \chi\Delta \end{pmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Sigma &= \hat{T}(R)[R+\hat{T}(R)]^{-1}[R+T(R)][T(R)]^{-1} \\
&= \begin{pmatrix} T(R_{11}) & 0 \\ 0 & \frac{\tilde{R}+\Delta}{2} \end{pmatrix} \begin{pmatrix} \mathrm{I} & \frac{1}{2}\Gamma R_{1,2} R_{2,2}^{-1} \Delta \\ 0 & \mathrm{I} - \chi\Delta \end{pmatrix} \begin{pmatrix} T(R_{11}) & 0 \\ 0 & \frac{\tilde{R}-\Delta}{2} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} T(R_{11}) & \frac{1}{2} T(R_{11}) \Gamma R_{1,2} R_{2,2}^{-1} \Delta \\ 0 & \frac{\tilde{R}+\Delta}{2}(\mathrm{I} - \chi\Delta) \end{pmatrix} \begin{pmatrix} [T(R_{11})]^{-1} & 0 \\ 0 & 2(\tilde{R}-\Delta)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{I} & T(R_{11}) \Gamma R_{1,2} R_{2,2}^{-1} \Delta (\tilde{R}-\Delta)^{-1} \\ 0 & (\tilde{R}+\Delta)(\mathrm{I}-\chi\Delta)(\tilde{R}-\Delta)^{-1} \end{pmatrix}.
\end{aligned}$$

First, note that $\Sigma_{\overline{\mathcal{M}},\overline{\mathcal{M}}} = \mathrm{I} \geq 0$.

Second, consider $\Sigma_{\overline{\mathcal{M}},\mathcal{N}\setminus\overline{\mathcal{M}}} = T(R_{11}) \Gamma R_{1,2} R_{2,2}^{-1} \Delta (\tilde{R} - \Delta)^{-1}$. Note that $R_{11} + T(R_{11}) - \frac{1}{2} R_{1,2} R_{2,2}^{-1} R_{2,1}$ is the Schur complement of a principal sub-matrix $2R_{2,2}$ of matrix $R + \hat{T}(R) = \begin{pmatrix} R_{11}+T(R_{11}) & R_{1,2} \\ R_{2,1} & 2R_{2,2} \end{pmatrix}$, which is symmetric since $R_{1,2} = R_{2,1}^{\top}$. Since $R + \hat{T}(R)$ is positive definite (both R and $\hat{T}(R)$ are positive definite with $\hat{T}(R)$ replacing some off-diagonal negative submatrices of R with 0), $R_{11}+T(R_{11})-\frac{1}{2}R_{1,2}R_{2,2}^{-1}R_{2,1}$ is positive definite as well. Since $R_{1,2}\le 0$ and $R_{2,1}\le 0$ (due to R is a Z-matrix) and $R_{2,2}^{-1}\ge 0$ (due to the fact that $R_{2,2}$ is a principal submatrix of a ZP-matrix R, is a ZP-matrix as well), $R_{11}+T(R_{11})-\frac{1}{2}R_{1,2}R_{2,2}^{-1}R_{2,1}\le R_{11}+T(R_{11})$. Since $R_{11}+T(R_{11})$ is a Z-matrix, $R_{11}+T(R_{11})-\frac{1}{2}R_{1,2}R_{2,2}^{-1}R_{2,1}$ is a Z-matrix as well. Therefore, $R_{11}+T(R_{11})-\frac{1}{2}R_{1,2}R_{2,2}^{-1}R_{2,1}$ is a ZP-matrix, and thus $\Gamma=\left(R_{11}+T(R_{11})-\frac{1}{2}R_{1,2}R_{2,2}^{-1}R_{2,1}\right)^{-1}\ge 0$.

Since $R_{11}+T(R_{11})-\frac{1}{2}R_{1,2}R_{2,2}^{-1}R_{2,1}\le R_{11}+T(R_{11})\le 2T(R_{11})$, post-multiplying by $\Gamma\ge 0$ gives:

$$0\le \mathrm{I}\le 2T(R_{11})\Gamma.$$

Moreover, $R_{1,2}\le 0$ (due to the fact that R is a Z-matrix), $R_{2,2}^{-1}\ge 0$ (due to $R_{2,2}$ being a ZP-matrix, see above), $\Delta\le 0$ (by definition) and $(\tilde{R}-\Delta)^{-1}\ge 0$ (due to $(\tilde{R}-\Delta)^{-1}=\frac{1}{2}\begin{pmatrix}R_{22} & 0\\ 0 & R_{33}\end{pmatrix}^{-1}=\begin{pmatrix}R_{22}^{-1} & 0\\ 0 & R_{33}^{-1}\end{pmatrix}\ge 0$ since both R_{22} and R_{33} are ZP-matrices). Thus, we reach the conclusion that $\Sigma_{\overline{\mathcal{M}},\mathcal{N}\setminus\overline{\mathcal{M}}}=T(R_{11})\Gamma R_{1,2}R_{2,2}^{-1}\Delta(\tilde{R}-\Delta)^{-1}\ge 0$. □

THEOREM 3. *Assume, in addition to intra-firm symmetry (Assumption 3), $R_{1,2}=R_{2,1}^{\top}$. For any $w\ge 0$ such that $[\hat{\Psi}(R)(a-R\Theta(w))]_{\mathcal{M}}\ge 0$,*[4] *we have*

$$\mathcal{N}^*(w)\subseteq\hat{\mathcal{N}}^*(w).$$

Proof of Theorem 3. The proof technique is the same as that of Theorem 1(ii).

The pre-merger equilibrium sales volumes are given by: $\Psi(R)(a-R(w-t^*))$, and the post-merger equilibrium volumes by: $\hat{\Psi}(R)(a-R(w-\hat{t}^*))$, where t^* is the unique solution to the LCP: $\Psi(R)(a-R(w-t))\ge 0$, $t^\top[\Psi(R)(a-R(w-t))]=0$ and $t\ge 0$, and $\hat{t}^*$ is the unique solution to LCP: $\hat{\Psi}(R)(a-R(w-\hat{t}))\ge 0$, $\hat{t}^\top[\hat{\Psi}(R)(a-R(w-\hat{t}))]=0$ and $\hat{t}\ge 0$.

To prove $\mathcal{N}^*(w)\subseteq\hat{\mathcal{N}}^*(w)$, it therefore suffices to show that

$$\hat{t}^*\le t^*. \tag{20}$$

By Mangasarian (1976, Theorem 3), the (unique) solution to the LCP: $\Psi(R)(a-R(w-t))\ge 0$, $t^\top[\Psi(R)(a-R(w-t))]=0$ and $t\ge 0$ is the (unique) solution of any LP of the form:

$$\min\ \pi^\top t \quad \text{s.t.}\ \Psi(R)(a-R(w-t))\ge 0,\quad t\ge 0, \tag{21}$$

[4] By Federgruen and Hu (2021, Eq. (6)), $\Psi(R)(a-R\Theta(w))$ is the equilibrium sales volume vector before the merger. So one can take that sales volume vector and pre-multiply it with $\hat{\Psi}(R)[\Psi(R)]^{-1}$ and check the entries in the set of $\mathcal{M}$.

where π is *any* vector of positive coefficients. Similarly, the (unique) solution to the LCP: $\hat{\Psi}(R)(a-R(w-\hat{t}))\geq 0$, $\hat{t}^\top[\hat{\Psi}(R)(a-R(w-\hat{t}))]=0$ and $\hat{t}\geq 0$ is the (unique) solution of any LP of the form:

$$\min\ \pi^\top\hat{t} \quad \text{s.t. } \hat{\Psi}(R)(a-R(w-\hat{t}))\geq 0, \quad \hat{t}\geq 0, \tag{22}$$

where π is *any* vector of positive coefficients.

Fix any l and set $\pi_l=1$ and $\pi_{l'}=\epsilon$ for any $l'\neq l$. Since $\Theta(w)=w-t^*\in W$, by Proposition 2, $[\hat{\Psi}(R)(a-R(w-t^*))]_{\overline{\mathcal{M}}}\geq 0$. By the stipulation, $[\hat{\Psi}(R)(a-R(w-t^*))]_{\mathcal{M}}\geq 0$. That is, the optimal solution t^* to (21) is a feasible solution to (22). Thus, $\hat{t}^*_l+\epsilon\sum_{l'\neq l}\hat{t}^*_{l'}\leq t^*_l+\epsilon\sum_{l'\neq l}t^*_{l'}$. Note that both $\hat{t}^*_{l'}\leq w_{l'}$ and $t^*_{l'}\leq w_{l'}$. Now we let $\epsilon\searrow 0$ on both sides of the inequality and conclude that $\hat{t}^*_l\leq t^*_l$. This applies for any $l=1,\ldots,I$. $\square$

Theorem 3 suggests that when the merged firm does not prefer to shrink its product assortment after the horizontal merger, the market equilibrium assortment would weakly expand because the less intense competition with higher equilibrium prices (see Theorem 2(i)) in the post-merger world would allow more products to survive in the market.

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